The Equations used in Comparing Electric Air Taxi Visions

By
Dr. David G. Ullman

The models used in “Comparing Electric Air Taxi Visions” are developed here.

Variables

\( \rho_a \) = air density (taken as sea level)
\( \rho_b \) = battery energy density of 150, 300, 450 and 600 wh/kg
\( \eta \) = total combined efficiency from batteries to useful thrust (.85)
AGL = cruise altitude
CL\text{\textsubscript{to}} = Take-off lift coefficient
ROC = Rate of climb
DL = Disk loading (lb/ft\(^2\))
Ec = Battery capacity (kwh)
\( E_h \) = however energy
\( L/D_{\text{climb}} \) = climb L/D ratio
\( L/D_{\text{cruise}} \) = cruise lift to drag ratio
Np = number of passengers
\( P_h \) = Power for hover (ft lb/sec)
\( R \) = Range is in miles
\( S_g \) = ground roll for take-off
\( t_c \) = time to climb (sec)
\( t_{cr} \) = cruise time (sec)
\( t_h \) = hover time (sec)
\( tr \) = reserve time (sec)
\( V_c \) = \( V_{\text{cruise}} \) = Cruise velocity (ft/sec)
\( V_{\text{to}} \) = take-off velocity
\( Wo \) = total weight (lb)
\( Wa \) = airframe weight including all electronics and motors, excluding batteries (lb)
\( Wbt \) = total battery weight (lb)
\( W_{\text{bt}} \) = battery weight for taxi and takeoff (lb)
\( W_{bh} \) = battery weight for hover (lb)
\( W_{bc} \) = battery weight for climb (lb)
\( W_{bcr} \) = battery weight for cruise (lb)
\( W_{bd} \) = battery weight for descent (lb)
\( W_{blt} \) = battery weight for landing and taxi (lb)
\( W_{br} \) = battery weight reserve (lb)
\( Wp \) = payload weight (assume 220 lb (100kg) per passenger.)
The Input And Calculated Variables.

Basic logic - for each combination of:
- Sky taxi class
- Battery energy density
- Battery capacity (energy stored)
- Number of passengers

And these assumptions (rational ad sensitivity given in paper):
- Cruise velocity (mph/fps/kts) $V_c=150/220/130$
- Cruise altitude (ft/m) $AGL = 1000/300$
- Climb rate (CR) 500 ft/min (2.5 m/sec) then time to climb (tc) is 2 minutes
- Electrical and propulsor efficiency = 0.76
- Airframe weight fraction including motors and controllers = 50%
- Passenger and luggage weight (kg/lb) = 100/220
- Factor of Merit (FOM) = 0.7
- Hover time (sec) = 60 (120 sec explored in sensitivity analysis)
- Reserve time (min) = 10
- Percent battery capacity available = 80%
- Battery cost ($/kwh): Current = $200, Near future = $100
- USTOL take-off distance (ft) = 150
- USTOL take-off velocity (mph/ft/sec) = 35/51

<table>
<thead>
<tr>
<th>Class</th>
<th>L/D Climb</th>
<th>L/D Cruise</th>
<th>Disk Loading (lb/ft²)</th>
<th>Thrust/weight Ratio</th>
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</thead>
<tbody>
<tr>
<td>Rotor</td>
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<td>4.25</td>
<td>4.5</td>
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<td>VTOL</td>
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<td>USTOL</td>
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<td>15</td>
<td>0.34</td>
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Follow these steps:
- Step 1: Based on the input values find the gross weight of the aircraft.
- Step 2: Calculate the battery weight fractions for taxi/take-off, hover, climb, descent, landing and reserve (note that the battery weight fraction for cruise is not included here).
- Step 3: Find the battery weight fraction for cruise.
- Step 4: Calculate the range.
- Step 5: Calculate the needed maximum thrust and power.
- Step 6: Estimate the aircraft unit cost.

Details for each step given below.

**Calculate Gross Weight (Step 1)**

Regardless of aircraft class, the gross weight (Wo) is composed of the weight of the airframe (Wa), total weight of the batteries (Wbt) and the weight of the passengers (Wp). Here, the airframe includes the motors, wiring and controllers, everything but the passengers and batteries.

\[
Wo = Wa + Wbt + Wp
\]  

1

This can be rewritten as weight fractions:

\[
1 = \frac{Wa}{Wo} + \frac{Wbt}{Wo} + \frac{Wp}{Wo} \quad 2
\]

or

\[
Wo = \frac{Wbt}{(1 – \frac{Wp}{Wo} – \frac{Wa}{Wo})} \quad 3
\]

The total weight is the battery weight divided by one minus the passenger weight fraction and the airframe weight fraction.
The Passenger Fraction

\[ \frac{W_{p}}{W_{o}} = \frac{N_{p} \times 220}{W_{o}} \]

Where \( N_{p} \) is the number of passengers including the pilot (if any) all assumed to weigh 220 lb (100 kg).

The Airframe Fraction

\( \frac{W_{a}}{W_{o}} \) is the airframe fraction. It includes everything but passengers and the batteries. Curve fitting multiple modern composite aircraft with the weight of their engines subtracted out gives an average value of .44. McDonald and German used .55. They were working with 5000 lb vehicles which may have a higher fraction. The X-57 Maxwell will have a gross weight of 3000, has 900 lbs of batteries and, with a crew of 2, 440 lb of payload. This gives an airframe weight fraction of .55. The .44 value is for lighter vehicles and does not include an allowance for the motors and wiring needed. Fitting these two points, then

\[ \frac{W_{a}}{W_{o}} = 0.366 \times (1 + 0.0001 \times W_{o}) \]

Battery weight

The total weight of the battery (lb) is a function of the energy density and the battery capacity:

\[ W_{bt} = 2200 \times \frac{E_{c}}{\rho_{b}} = 2.2 \times N_{p} \times E_{c}' \]  
\[ (2200 = 2.2 \text{ (lb/kg)} \times 1000 \text{ (w/kw)}) \]

with \( E_{c}' = 1000 \times \frac{E_{c}}{(N_{p} \times \rho_{b})} \) the battery weight (kg) per passenger.

Gross weight

We now use all the parts to find the gross weight:

\[ W_{o} = \frac{W_{bt}}{(1 - \frac{W_{p}}{W_{o}} - \frac{W_{a}}{W_{o}})} \]

Or

\[ W_{o} = 2.2 \times N_{p} \times \frac{(E_{c}' + 100)}{(1 - \frac{W_{a}}{W_{o}})} \]

Can also find the total battery fraction:

\[ \frac{W_{bt}}{W_{o}} = (1 - \frac{W_{a}}{W_{o}}) \times \frac{(E_{c}' / (E_{c}' + 100))} \]

These two equations are critical to the model.
Battery Weight Fractions (Step 2)
With the total weight fraction known, the goal is to find the cruise battery weight or fraction (in bold) based on subtracting the battery weight fraction for all the other segments of the entire mission from the total. In general:

$$\frac{W_{bt}}{W_o} = \frac{W_{bto}}{W_o} + \frac{W_{bh}}{W_o} + \frac{W_{bc}}{W_o} + \frac{W_{bcr}}{W_o} + \frac{W_{bd}}{W_o} + \frac{W_{blt}}{W_o} + \frac{W_{br}}{W_o}$$

So:

$$\frac{W_{bc}}{W_o} = \frac{W_{bt}}{W_o} - (\frac{W_{bto}}{W_o} + \frac{W_{bh}}{W_o} + \frac{W_{bcr}}{W_o} + \frac{W_{bd}}{W_o} + \frac{W_{blt}}{W_o} + \frac{W_{br}}{W_o})$$

In the next sections the battery weight fractions are estimated in the order they occur.

Battery Weight Fraction for Taxi and Take-Off
For VTOL and rotor-craft it is assumed there is no taxi and take-off power used, so there is no battery weight fraction for them.

For USTOL power is used for taxi and take off includes acceleration, aero drag, and wheel friction. For USTOL with short take-offs, by far the largest of these is the power used to accelerate the mass of the vehicle. This is directly a function of the desired take-off distance and velocity needed. See the discussion in Step 5 for more on this.

For here, the battery fraction used during this phase is very small, < 1% of the total battery so, for all models:

$$\frac{W_{bto}}{W_o} = 0.0$$

Battery Weight Fraction for Hover
For a helicopter the power (ft lb/sec) can be found$^1$ (this will be used again later). First, based on induced velocity through the rotor, the power loss in hover is:

$$P_h = \frac{T^{3/2}}{(2 \cdot \rho_a \cdot A)^{1/2}}$$

where $A$ is the rotor area.

With thrust = weight ($W_o$) and the disk loading ($DL$) = weight divided by the area ($W_o/A$), this can be rewritten as

$$P_h = W_o \cdot (DL / (2 \cdot \rho_a))^{1/2}$$

Additionally, for rotor craft a figure of merit must included (FOM). This gives the rotor efficiency and is typically about .7, so on a standard day

$$P_h = 14.5 \cdot W_o \cdot DL^{.5} \cdot FOM \text{ (ft lb/sec)}$$

$^1$ W. Johnson, Helicopter Theory, Dover, 1994, pg 31
Energy for hover (ft lb) is the power times the hover time ($t_h$).

$$E_h = 14.5 \times t_h \times Wo \times DL^{.5} \times FOM \quad 16$$

Or, in watt hours:

$$E_h = 0.0056 \times t_h \times Wo \times DL^{.5} \times FOM \quad 17$$

With battery energy density $\rho_b$, the battery weight for hover (lbs) is:

$$W_{bh} = 0.0056 \times t_h \times Wo \times DL^{.5} \times FOM / \rho_b \quad 18$$

Or for weight fraction

$$W_{bh}/Wo = 0.0056 \times t_h \times DL^{.5} \times FOM / \rho_b \quad 19$$

**Battery Weight Fraction for Climb and Descent**

Regardless of the type of vehicle, there is the increase in potential and kinetic energy during climb. The potential energy is just the weight ($Wo$) times the altitude change and the kinetic energy is the one half the mass times the velocity squared. With the efficiency $\eta$, then battery energy needed to climb is:

$$E_{cl} = (Wo \times AGL + .5 \times Wo/ 32.2 \times V_c^2) / \eta \quad 20$$

Or

$$E_{cl} = Wo \times (AGL + .0155 \times V_c^2 / AGL) / \eta$$

With the assumed values for altitude (AGL =1000 ft), velocity (Vc=150/220/130 (mph/fps/kts) )and efficiency ($\eta = .76$) ; this reduces to:

$$E_{cl} = Wo \times 2302 \text{ (ft lb)} \quad 21$$

With the rate of climb (ROC = 500 ft/min) assumed, the time to climb ($t_c$) can be found and then the power needed is:

$$P_{cl1} = 0.0167 \times ROC \times Wo \times (1 +.0155 \times V_c^2 / AGL) / \eta \quad 22$$

Or with the assumed values:

$$P_{cl1} = 19.2 \times Wo \text{ (ft lb/sec)} \quad 23$$

Power used to overcome aerodynamic drag is (assumed an average velocity):

$$P_{cl2} = D \times Vavg = Wo \times Vavg / (L/Dc) \quad 24$$

Or with assumed values

$$P_{cl2} = 110 \times Wo \times (L/Dc) \text{ (ft lb/sec)} \quad 25$$
So the total power for climb is

\[ P_{cl} = Wo \times \left( \frac{110}{(L/D) + 19.2} \right) \text{ (ft lb/sec)} \]

Energy is \( P \times t \) where \( t = 120 \) sec

\[ E_{cl} = 120 \times Wo \times \left( \frac{110}{(L/D)_{climb}} + 19.2 \right) \text{ ftlb} \]

\[ E_{cl} = 0.045 \times Wo \times \left( 110/(L/D_{climb}) + 19.2 \right) \text{ (wh)} \]

So battery weight (lbs)

\[ B_{cl} = 0.099 \times Wo \times \left( \frac{110}{(L/D)_{climb}} + 19.2 \right) / \rho_b \]

In terms of weight fraction

\[ B_{cl}/Wo = 0.099 \times \left( \frac{110}{(L/D)_{climb}} + 19.2 \right) / \rho_b \]

Distance for climb

\[ R_{to} = 0.5 \times t^2 \times \left( \frac{220}{120} \right) / 5280 = 2.5 \text{ miles} \]

For descent the formula is the same but KE and potential energy are recovered by changing the sign of the second term, so:

\[ B_{d} = 0.097 \times Wo \times \left( \frac{110}{(L/D)} - 19.2 \right) / \rho_b \]

In terms of weight fraction

\[ B_{cl}/Wo = 0.097 \times \left( \frac{110}{(L/D)} - 19.2 \right) / \rho_b \]

**Battery Weight Fraction for Reserve**

The additional range for reserve can either be given as a distance or a time. Here, it will be a time (\( t_r \)) assumed at cruise velocity (\( V_c \)) as if traversing to an alternative airport.

So the reserve range (miles) is:

\[ R_{reserve} = t_r \times V_cruise \times 60/5280 \]

Then, using the formula for range as a function of cruise battery fraction, rearrange to get

\[ \frac{W_{br}}{Wo} = \frac{R_{reserve}}{0.227 \times \rho_b \times (L/D_{cruise}) \times \eta} \]

\[ \frac{W_{br}}{Wo} = 0.050 \times t_r \times V_{cruise} / (\rho_b \times L/D_{cruise} \times \eta) \]
Weight Fraction for Cruise (Step 3)
Since the total battery weight fraction is known from Step 1 and the battery weight fractions for all mission phases except cruise are found in Step 2, the difference is the cruise battery fraction. If positive there is sufficient battery for time at the cruise altitude and velocity. If negative, then the mission cannot be accomplished. With the total battery weight fraction at

\[
\frac{W_{bt}}{W_0} = \frac{W_{bto}}{W_0} + \frac{W_{bh}}{W_0} + \frac{W_{bc}}{W_0} + \frac{W_{bcr}}{W_0} + \frac{W_{bd}}{W_0} + \frac{W_{blt}}{W_0} + \frac{W_{br}}{W_0}
\]

Then:

\[
\frac{W_{bc}}{W_0} = (1 - \frac{W_a}{W_0}) \times \frac{E_{c'}}{(E_{c'} + 100)} - (\frac{W_{bto}}{W_0} + \frac{W_{bh}}{W_0} + \frac{W_{bcr}}{W_0} + \frac{W_{bd}}{W_0} + \frac{W_{blt}}{W_0} + \frac{W_{br}}{W_0})
\]

Range (Step 4)
The range is the sum of the climb, descent and cruise ranges. The climb and descent ranges were computed above based on the cruise velocity and time to climb. As mentioned earlier, a 500 ft/min climb rate has been assumed and, with the cruise velocity of 150 mph, the distance covered during climb and descent is 2.5 miles each for all classes of sky taxis.

The cruise range model is built around Breuget's equation for electric airplanes. This was developed by the author for a class in in 2014 and is similar to Hepperle’s equation\(^2\).

- Power required = Power available.
- Power required = \(\text{Drag} \times \text{Velocity} = \frac{W_0 \times V}{(L/D_{\text{cruise}})}\)
- Power available = \(W_{bcr} \times \rho_b \times \eta / t_c\)

Based on the battery energy available for cruise.

Equating and realizing that battery energy density is effectively distance (1 wh = 2655 ft
\(\text{lb} = 367 \text{ kg m}\):

\[
1 \text{ wh/kg} = .367 \text{ km} = 1204 \text{ ft}
\]

Then

\[
t_c = .367 \times \frac{W_{bcr} \times \rho_b \times \eta \times (L/D_{\text{cruise}})}{(W_0 \times V)}
\]

With Range is \(V_{cr} \times t_c\)

\[ R_{\text{cruise}} \text{ (km)} = .367 \times (W_{\text{bcr}}/W_{\text{o}}) \times \rho_b \times (L/D_{\text{cruise}}) \times \eta \]

or

\[ R_{\text{cruise}} \text{ (miles)} = .227 \times (W_{\text{bcr}}/W_{\text{o}}) \times \rho_b \times (L/D_{\text{cruise}}) \times \eta \]

So total range is:

\[ R = 2 \times R_{\text{climb}} + R_{\text{cruise}} \]

**Thrust and Power Required (Step 5)**

**Thrust**

The needed maximum thrust is a direct function of weight for all classes of vehicle. It is substantially less for USTOL than for rotor-craft and VTOL as explained in the assumptions.

For VTOL and rotors (helicopters)

\[ T/W_{\text{o}} = 1.25 \text{ (discussed in the paper)} \]

For USTOL it is more complex. The thrust required is the sum of that required for mass acceleration, aerodynamic drag and wheel friction. For these short takeoffs at low speeds the acceleration of the mass is dominant. The acceleration is proportional to the thrust to weight ratio:

\[ T/W_{\text{o}} = a/g \text{ where } a \text{ is the acceleration of the aircraft (assumed constant) and } g \text{ is the acceleration of gravity.} \]

Or

\[ a = T/W_{\text{o}} \times g \]

From basic mechanics, the distance for acceleration (i.e. the ground roll for take-off) is:

\[ S_g = V_{\text{to}}^2 / (2 \times a) \]

Or

\[ S_g = V_{\text{to}}^2 / (2 \times T/W \times g) \]

The velocity for takeoff is when the lift equals the weight, or:

\[ V_{\text{to}}^2 = 2 \times W_{\text{o}} / (\rho_a \times C_{\text{L_{to}}} \times A) \]

So

\[ S_g = W_{\text{o}} / (\rho_a \times g \times C_{\text{L_{to}}} \times A \times T/W) \]
Rearranging then the needed thrust is:

\[ T = \frac{W_0^2}{\left(\rho_a * g * C_{L_{to}} * A * S_g\right)} \]

Alternatively, based on 49 and 50

\[ \frac{T}{W_0} = \frac{V_{to}^2}{\left(2 * S_g * g\right)} \]

Or, with the assumed values for takeoff distance and speed:

\[ \frac{T}{W_0} = 0.27 \text{ (USTOL)} \]

Also, based on 52

\[ C_{L_{to}} * A = 2 * \frac{W_0}{\left(\rho_a * V_{to}^2\right)} \]

The term \((C_{L_{to}} * A)\) is controlled by the size of the lifting surfaces and the lift generated by them. For PAIs, take-off lift coefficients of 6 and above are possible. Further the lifting surface area can be controlled. Thus the term \(C_{L_{to}} * A\) could be treated as an input variable, and the takeoff velocity calculated. For good performance, it should be equal to one third the weight \((\text{CLA} = \frac{W_0}{3})\) of the aircraft. Thus an LSA sized craft where \(W_0 = 1320\text{lbs}\), a wing area of 100sqft and a takeoff velocity of 51ft/sec then CL at takeoff would need to be 4.3, not unrealistic for a USTOL.

For comparison purposes, the takeoff distance was set at 150 ft. This is seen as adequately short for a pocket airport. Note that it is shorter than the distance for a Carbon Cub, a very high performance LSA STOL.

**Power**

The maximum power for VTOL and rotor-craft is that used to hover and is a function of the weight, Figure of Merit (FOM) and Disk Loading (DL). For USTOL the maximum power required is that needed for take-off acceleration. The shorter the take-off run, the more power needed.

Power for VTOL and rotor craft is (from the Battery Weight Fraction for Hover)

\[ P_h = 14.5 * \frac{W_0 * DL ^{0.5} * \text{FOM}}{} \text{ (ft lb/sec)} \]

For USTOL the power can be found for that needed by the ground run distance \((S_g)\) and the take off velocity \((V_{to})\). Specifically:

\[ P = \text{force} * V_{to} = \text{mass} * \text{acceleration} * V_{to} \]

Form basic mechanics, acceleration = \(V_{to}^2 / (2 * S_g)\), so:

\[ P = \frac{W_0 * g * V_{to}^2}{2 * S_g} \]
Aircraft cost (Step 6)
The aircraft cost is composed of the cost of the airframe and the cost of the batteries. The cost of the motors, controllers and wiring are assumed included in the airframe. Since all the classes of aircraft considered will be error to the same degree, this assumption is sufficient. To find the cost only

- N (production number) 1000
- Labor rate/hr $60/hr
- battery cost ($/kwh) Current 200$/kwh, Near Future 100$/kwh

Manufacturing hours (Eqn 2.4 in Gudmundsson³)

\[ \text{MfgHrs} = 9.66 \times W_0^{0.74} \times V_{\text{cruise}}^{0.54} \times N^{0.524} \times 1.25 \]

Manufacturing Hours (Eqn 2.90)

\[ \text{MfgCost} = 2.1 \times \text{MfgHrs} \times \text{LaborRate} \]

Material cost (Eqn 2.11)

\[ \text{MatCost} = 24.9 \times W_0^{0.69} \times V_{\text{cruise}}^{0.62} \times N^{0.79} \]

Battery cost

\[ \text{BatCost} = B_{\text{kwh}} \times \text{BC} \]

Total cost = MfgCost + MatCost + BatCost